[Maximum mark: 6] 20160608

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ is convergent or divergent.

[Maximum mark: 9]

(a) Using l'Hopital's Rule, show that $\lim_{x\to\infty} xe^{-x} = 0$. *[2 marks]*

(b) Determine $\int_0^a xe^{-x}\,dx$. *[5 marks]*

(c) Show that the integral $\int_0^\infty xe^{-x}\,dx$ is convergent and find its value. *[2 marks]*

[Maximum mark: 5]

The real root of the equation $x^3 - x + 4 = 0$ is -1.796 to three decimal places. Determine the real root for each of the following.

(a) $(x-1)^3 - (x-1) + 4 = 0$ *[2 marks]*

(b) $8x^3 - 2x + 4 = 0$ *[3 marks]*

[Maximum mark: 5]

The wingspans of a certain species of bird can be modelled by a normal distribution with mean 60.2 cm and standard deviation 2.4 cm.

According to this model, 99 % of wingspans are greater than x cm.

(a) Find the value of x. *[2]*

In a field experiment, a research team studies a large sample of these birds. The wingspans of each bird are measured correct to the nearest 0.1 cm.

(b) Find the probability that a randomly selected bird has a wingspan measured as 60.2 cm. *[3]*

[Maximum mark: 6]

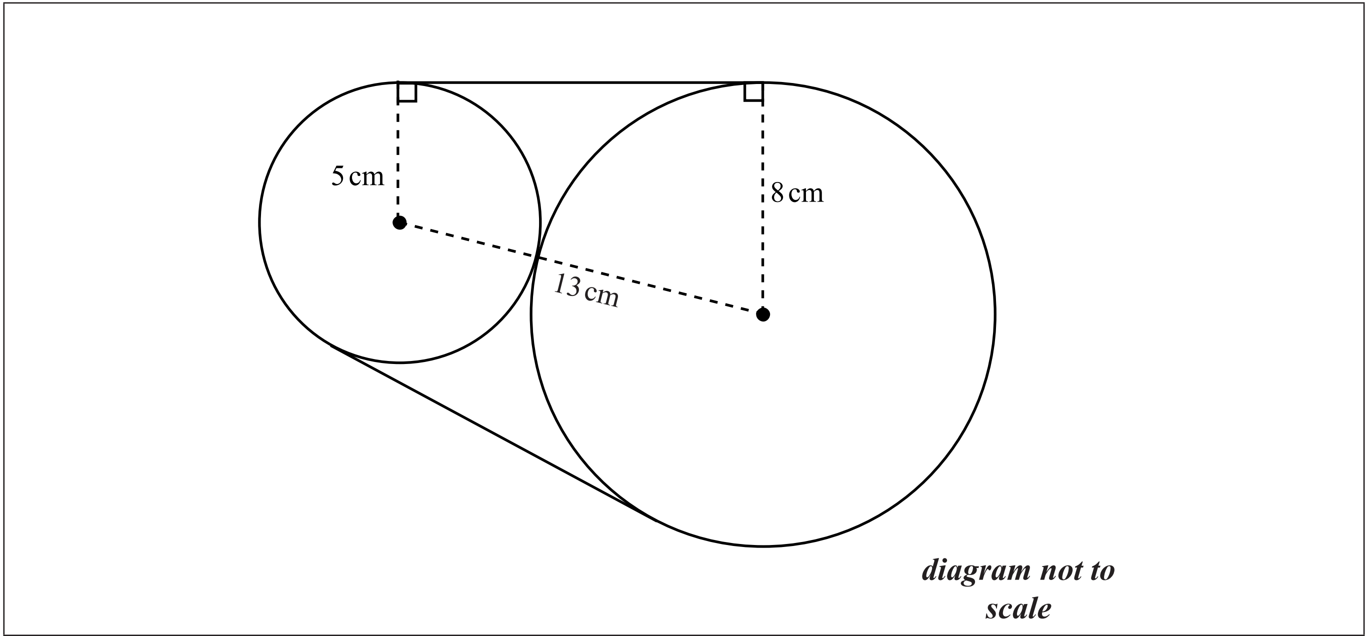
A function f is defined by $f(x)=\dfrac{2x-3}{x-1}$, $x \neq 1$.

(a) Find an expression for $f^{-1}(x)$. *[3 marks]*

(b) Solve the equation $\left|f^{-1}(x)\right|=1+f^{-1}(x)$. *[3 marks]*

[Maximum mark: 8]

Two discs, one of radius 8 cm and one of radius 5 cm, are placed such that they touch each other. A piece of string is wrapped around the discs. This is shown in the diagram below.



Calculate the length of string needed to go around the discs.

[Maximum mark: 16]

Consider $\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$. *[5 marks]*

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \frac{\pi}{2}$ on an Argand diagram. *[4 marks]*

(c) (i) Expand and simplify $F(z) = (z-1)(z-\omega)(z-\omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω. *[7 marks]*

[Maximum mark: 22]

A particle moves in a straight line with velocity v metres per second. At any time t seconds, $0 \le t < \frac{3\pi}{4}$, the velocity is given by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} + v^2 + 1 = 0$. It is also given that $v = 1$ when $t = 0$.

(a) Find an expression for v in terms of t. *[7 marks]*

(b) Sketch the graph of v against t, clearly showing the coordinates of any intercepts, and the equations of any asymptotes. *[3 marks]*

(c) (i) Write down the time T at which the velocity is zero.

(ii) Find the distance travelled in the interval $[0, T]$. *[3 marks]*

(d) Find an expression for s, the displacement, in terms of t, given that $s = 0$ when $t = 0$. *[5 marks]*

(e) Hence, or otherwise, show that $s = \frac{1}{2}\ln\frac{2}{1+v^2}$. *[4 marks]*

[Maximum mark: 17]

The function f is defined by

$$f(x) = \ln\left(\frac{1}{1-x}\right).$$

(a) Write down the value of the constant term in the Maclaurin series for $f(x)$. *[1 mark]*

(b) Find the first three derivatives of $f(x)$ and hence show that the Maclaurin series for $f(x)$ up to and including the x^3 term is $x + \frac{x^2}{2} + \frac{x^3}{3}$. *[6 marks]*

(c) Use this series to find an approximate value for $\ln 2$. *[3 marks]*

(d) Use the Lagrange form of the remainder to find an upper bound for the error in this approximation. *[5 marks]*

(e) How good is this upper bound as an estimate for the actual error? *[2 marks]*

[Maximum mark: 6]

If x satisfies the equation $\sin\left(x+\frac{\pi}{3}\right)=2\sin x\sin\left(\frac{\pi}{3}\right)$, show that $11\tan x = a + b\sqrt{3}$, where a, $b \in \mathbb{Z}^{+}$.

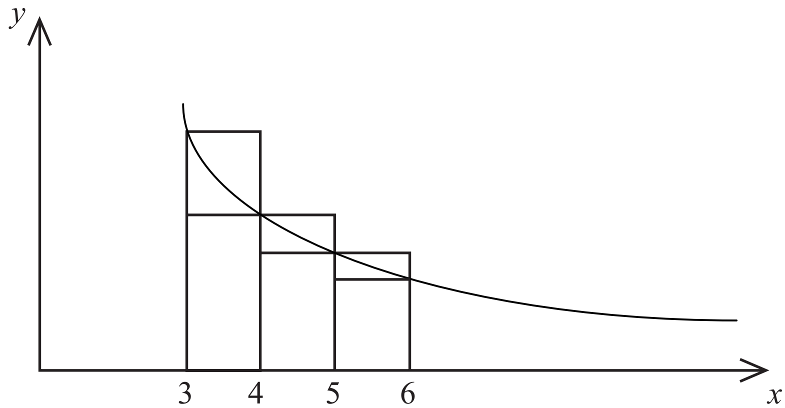
[Maximum mark: 7]

The sum of the first 16 terms of an arithmetic sequence is 212 and the fifth term is 8.

(a) Find the first term and the common difference. *[4 marks]*

(b) Find the smallest value of n such that the sum of the first n terms is greater than 600. *[3 marks]*

[Maximum mark: 15]



The diagram shows part of the graph of $y = \dfrac{1}{x^3}$ together with line segments parallel to the coordinate axes.

(a) Using the diagram, show that

$$\frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \ldots < \int_3^\infty \frac{1}{x^3}\,\mathrm{d}x < \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots .$$

[3 marks]

(b) **Hence** find upper and lower bounds for $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

[12 marks]